Initial fluctuations (for Lemma 4.3)

In[1]:= $\texttt{pi[j\_]} := E^{-c}\, c^{j} / j!$

In[2]:= $\texttt{cov[j\_, k\_]} := \texttt{pi[j]} \times \texttt{pi[k]} \left(\frac{(j - c)(k - c)}{c} - 1 \right)$

In[3]:=
$$\Sigma_{1,1} = \texttt{cov[1, 1]} + \texttt{pi[1]};$$
$$\Sigma_{1,2} = \Sigma_{2,1} = \texttt{Sum[cov[1, k], \{k, 2, }\infty\texttt{\}]};$$
$$\Sigma_{1,3} = \Sigma_{3,1} = \texttt{Sum}\left[(k/2)\,\texttt{cov[1, k], \{k, 0, }\infty\texttt{\}}\right] + \texttt{pi[1]}/2;$$
$$\Sigma_{2,2} = \texttt{Sum[cov[j, k], \{j, 2, }\infty\texttt{\}, \{k, 2, }\infty\texttt{\}]} + \texttt{Sum[pi[j], \{j, 2, }\infty\texttt{\}]};$$
$$\Sigma_{2,3} = \Sigma_{3,2} = \texttt{Sum}\left[(k/2)\,\texttt{cov[j, k], \{j, 2, }\infty\texttt{\}, \{k, 0, }\infty\texttt{\}}\right] + \texttt{Sum}\left[(j/2)\,\texttt{pi[j], \{j, 2, }\infty\texttt{\}}\right];$$
$$\Sigma_{3,3} = \texttt{Sum}\left[(j/2)(k/2)\,\texttt{cov[j, k], \{j, 0, }\infty\texttt{\}, \{k, 0, }\infty\texttt{\}}\right] + \texttt{Sum}\left[(j/2)^2\,\texttt{pi[j], \{j, 0, }\infty\texttt{\}}\right];$$

In[9]:= $\texttt{Table[}\Sigma_{i,j}\texttt{, \{i, 3\}, \{j, 3\}] // Simplify // MatrixForm}$

Out[9]//MatrixForm=

$$\begin{pmatrix} c\, e^{-2c}\left(1 - 3c + c^2 + e^{c}\right) & -c\, e^{-2c}\left(-1 - 2c + c^2 + e^{c}\right) & -\left((-1 + c)\, c\, e^{-c}\right) \\ -c\, e^{-2c}\left(-1 - 2c + c^2 + e^{c}\right) & e^{-2c}\left(-1 - c^2 + c^3 + e^{c} + c\,(-2 + e^{c})\right) & c^2 e^{-c} \\ -\left((-1 + c)\, c\, e^{-c}\right) & c^2 e^{-c} & \frac{c}{2} \end{pmatrix}$$

Estimating the fluid limit (for Section 5.1)

In[10]:= $\texttt{f[u\_]} := E^{u} - u - 1$

In[11]:=
$$F_1\texttt{[x1\_, x2\_, x3\_]} := -1 - \frac{x1}{2\,x3} + \frac{x2^2\, \texttt{z[x1, x2, x3]}^4\, E^{\texttt{z[x1,x2,x3]}}}{\left(2\,x3\, \texttt{f[z[x1, x2, x3]]}\right)^2} - \frac{x1\, x2\, \texttt{z[x1, x2, x3]}^2\, E^{\texttt{z[x1,x2,x3]}}}{(2\,x3)^2\, \texttt{f[z[x1, x2, x3]]}};$$
$$F_2\texttt{[x1\_, x2\_, x3\_]} := -1 + \frac{x1}{2\,x3} - \frac{x2^2\, \texttt{z[x1, x2, x3]}^4\, E^{\texttt{z[x1,x2,x3]}}}{\left(2\,x3\, \texttt{f[z[x1, x2, x3]]}\right)^2};$$
$$F_3\texttt{[x1\_, x2\_, x3\_]} := -1 - \frac{x2\, \texttt{z[x1, x2, x3]}^2\, E^{\texttt{z[x1,x2,x3]}}}{2\,x3\, \texttt{f[z[x1, x2, x3]]}};$$

In[14]:= $\beta\texttt{[z\_]} := \texttt{ProductLog}\left[c\, E^{z}\right] / c$

(The Mathematica command for the Lambert W function is "`ProductLog`".)

In[15]:=
$$\texttt{chi}_1\texttt{[z\_]} := \frac{1}{c}\left(z^2 - z\, c\, \beta\texttt{[z]}\,(1 - E^{-z})\right);$$
$$\texttt{chi}_2\texttt{[z\_]} := \left(1 - (1 + z)\, E^{-z}\right) \beta\texttt{[z]};$$
$$\texttt{chi}_3\texttt{[z\_]} := \frac{1}{2\,c}\, z^2;$$

Now, let's re-express the drift function in terms of `z`:

In[18]:=
```
F[z_] := {F1[x1, x2, x3], F2[x1, x2, x3], F3[x1, x2, x3]} /.
  {z[x1, x2, x3] → z, x1 → chi1[z], x2 → chi2[z], x3 → chi3[z]}
```

Let’ s do a series expansion (Fact 5.2):

In[19]:=
```
Series[F[z], {z, 0, 0}] // MatrixForm
```

Out[19]//MatrixForm=

$$\begin{pmatrix} 2\,(-1+\text{ProductLog}[c]^2)+O[z]^1 \\ (-\text{ProductLog}[c]-\text{ProductLog}[c]^2)+O[z]^1 \\ (-1-\text{ProductLog}[c])+O[z]^1 \end{pmatrix}$$

Computing ∂F (for Lemma 5.3)

We will need to use implicit differentiation to compute $\partial z / \partial x_i$, so we introduce a quantity Q, which according to the definition of z can be expressed either in terms of z or x .

In[20]:=

$$Q[z\_] := \frac{(-1+e^z)\,z}{f[z]}$$

$$Q[x1\_,\, x2\_,\, x3\_] := \frac{-x1+2\,x3}{x2}$$

(Note that $\partial z / \partial x_i = \partial Q / \partial x_i \cdot dz / dQ$, and dz / dQ is the reciprocal of Q '[z].)

In[22]:=
```
dF = Table[
  (D[Fi[x1, x2, x3], xj] /.
    {z^(1,0,0)[x1, x2, x3] → Q^(1,0,0)[x1, x2, x3] / Q '[z[x1, x2, x3]],
     z^(0,1,0)[x1, x2, x3] → Q^(0,1,0)[x1, x2, x3] / Q '[z[x1, x2, x3]],
     z^(0,0,1)[x1, x2, x3] → Q^(0,0,1)[x1, x2, x3] / Q '[z[x1, x2, x3]],
     z[x1, x2, x3] → z, x1 → chi1[z], x2 → chi2[z], x3 → chi3[z]}),
  {i, 3}, {j, 3}];
```

Now let’s rewrite in terms of z, and do a series expansion around zero.

In[23]:=
```
Series[dF /. {z[x1, x2, x3] → z, x1 → chi1[z], x2 → chi2[z], x3 → chi3[z]}, {z, 0, -2}] /.
  {ProductLog → W} // MatrixForm
```

Out[23]//MatrixForm=

$$\begin{pmatrix} \frac{3\,c-7\,c\,W[c]}{z^2}+\frac{1}{O[z]} & -\frac{6\,(-c+c\,W[c])}{z^2}+\frac{1}{O[z]} & -\frac{2\,(3\,c-7\,c\,W[c]+4\,c\,W[c]^2)}{z^2}+\frac{1}{O[z]} \\ \frac{c+2\,c\,W[c]}{z^2}+\frac{1}{O[z]} & \frac{1}{O[z]} & \frac{2\,(-c-c\,W[c]+2\,c\,W[c]^2)}{z^2}+\frac{1}{O[z]} \\ \frac{4\,c}{z^2}+\frac{1}{O[z]} & \frac{6\,c}{z^2}+\frac{1}{O[z]} & \frac{-8\,c+2\,c\,W[c]}{z^2}+\frac{1}{O[z]} \end{pmatrix}$$

Next we want to diagonalise this matrix for Lemma 5.3 . Let’ s first rescale by z^2 and get rid of the error terms.

In[24]:=
```
H = Limit[z^2 %, z → 0];
```

In[25]:=
```
H // Eigenvectors // Transpose // MatrixForm
```

Out[25]//MatrixForm=

$$\begin{pmatrix} -2\,(-1+W[c]) & \frac{1}{2}\left(1-4\,W[c]+3\,W[c]^2\right) & -4\,W[c] \\ W[c] & \frac{1}{2}\left(1+W[c]-2\,W[c]^2\right) & 1+2\,W[c] \\ 1 & 1 & 1 \end{pmatrix}$$

(This is the matrix Q in Lemma 5.3.)

In[26]:=
```
DiagonalMatrix[H // Eigenvalues] // MatrixForm
```

Out[26]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3\,c\,(1+W[c]) & 0 \\ 0 & 0 & -2\,c\,(1+W[c]) \end{pmatrix}$$

(This is the matrix D in Lemma 5.3.)