The random k-matching-free process

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Abstract

Let $\mathcal{P}$ be a graph property which is preserved by removal of edges, and consider the random graph process that starts with the empty n-vertex graph and then adds edges one-by-one, each chosen uniformly at random subject to the constraint that $\mathcal{P}$ is not violated. These types of random processes have been the subject of extensive research over the last 20 years, having striking applications in extremal combinatorics, and leading to the discovery of important probabilistic tools. In this paper we consider the *k-matching-free process*, where $\mathcal{P}$ is the property of not containing a matching of size k. We are able to analyse the behaviour of this process for a wide range of values of k; in particular we prove that if $k = o(n)$ or if $n - 2k = o(\sqrt{n}/\log n)$ then this process is likely to terminate in a k-matching-free graph with the maximum possible number of edges, as characterised by Erdős and Gallai. We also show that these bounds on k are essentially best possible, and we make a first step towards understanding the behaviour of the process in the intermediate regime.

1 Introduction

Following Erdős and Rényi's seminal papers on random graphs [15, 16], there has been great interest in many different kinds of random graphs and random graph processes, with broad applications to various combinatorial problems and to real-world networks. The most basic random graph process, introduced by Erdős and Rényi, starts with the empty n-vertex graph and adds edges one-by-one, each selected uniformly at random among the edges not used so far. A particularly important variation of this basic process is the *random greedy* process. Here a decreasing[1] property $\mathcal{P}$ is specified, and then edges are added to the empty n-vertex graph one-by-one, chosen uniformly at random among edges whose addition to the current graph would not violate $\mathcal{P}$. A specific example of this type of process was first studied by Ruciński and Wormald [32] in 1992, and the idea was first discussed in full generality by Erdős, Suen and Winkler [17] in 1995.

Since then, a wide range of different types of random greedy processes have been studied. Perhaps the most famous specific example is the *triangle-free process*, where $\mathcal{P}$ is the property that a graph does not contain a triangle (see for example [17, 3, 19]). More generally, much of the work on random greedy processes has focused on cases of the *H-free process*, where $\mathcal{P}$ is the property that a graph does not contain a copy of a specified graph H (see for example [12, 30, 35, 7, 31, 34]). The theory

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[1] We say a graph property is *decreasing* if it is preserved by removal of edges, and we say a property is *increasing* if it is preserved by addition of edges.

of H-free processes has also been extended to hypergraphs (see for example [22, 6, 5, 27, 9]). We remark that in all the aforementioned results H is a fixed "small" (hyper)graph, whose size does not depend on n, and therefore the property of being H-free is in some sense a "local" constraint. Much less is known about random greedy processes for more "global" properties $\mathcal{P}$; two notable exceptions are the random greedy *planar graph process* [20], and the random greedy *k-colourable process* [17, 26].

There are a variety of different questions one can ask about random greedy processes. Commonly, one asks about the size and structure of the final (or almost-final) outcome of such a process. The process may a.a.s.[2] "saturate" and result in a graph with (almost) the maximum possible number of edges permitted by $\mathcal{P}$, or it may a.a.s. result in a graph with special properties that are useful for applications. Examples of the former situation include the *bounded-degree process* [32] that pioneered the study of random greedy processes, and the *triangle removal process* [5], which has become an important tool in the study of Steiner triple systems [25, 28]. A celebrated example of the latter situation is the triangle-free process, which a.a.s. produces triangle-free graphs with no large independent set; analysis of this process led to important breakthroughs in Ramsey theory [3, 19, 8]. There are also situations where the *intermediate* states of a random greedy process are of particular interest; for example, the intermediate stages of the random *satisfiable process* [26] are a good source of satisfiable formulas with certain unique properties.

In this paper we take a first look at the behaviour of the H-free process for an important choice of H with non-fixed size. A *k-matching* is a union of k disjoint edges. For any k (which may depend on n), the *k-matching-free process* is formally defined as follows. Let $N = \binom{n}{2}$, and let $e(1), \dots, e(N)$ be a uniformly random ordering of the unordered pairs in $\binom{[n]}{2}$ (that is, a random ordering of the edges of the complete graph K_n). This is the distribution obtained by iteratively selecting each $e(t)$ uniformly at random from the previously unseen edges. Let $G(0)$ be the empty n-vertex graph, and for $1 \le t \le N$ define

$$G(t) = \begin{cases} G(t-1) & \text{if } G(t-1) + e(t) \text{ contains a } k\text{-matching;} \\ G(t-1) + e(t) & \text{otherwise.} \end{cases}$$

In the former case we say $e(t)$ is *rejected* and in the latter case we say it is *accepted*. The outcome of this random process is a k-matching-free graph $G(N)$ which is k-matching-*saturated*, meaning that the addition of any edge would create a k-matching. We remark that the general notion of saturation in graphs and hypergraphs is of broad interest; see for example the surveys of Bollobás [10, Section 3] and Faudree, Faudree and Schmitt [18].

The general problem of determining whether a graph is k-matching-free (or, basically equivalently, the problem of determining the size of the largest matching in a graph) is of broad importance in various different areas of mathematics, computer science and even computational chemistry. One of the most basic results in this area is due to Erdős and Gallai [14], who proved that the maximum possible number of edges in a k-matching-free n-vertex graph is

$$\max\left\{ \binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1) \right\}.$$

This result falls under the umbrella of *extremal graph theory*, one of the central branches of modern combinatorics (see for example the book of Bollobás [11]). Up to isomorphism, the extremal graphs that attain the Erdős-Gallai bound are as follows.

[2]By "asymptotically almost surely", or "a.a.s.", we mean that the probability of an event is $1 - o(1)$. Here and for the rest of the paper, asymptotics are as $n \to \infty$.

- G_{clique} is a clique on $2k-1$ vertices with the remaining $n-2k+1$ vertices isolated.
- G_{star} is a clique on $k-1$ vertices, in addition to every possible edge between this clique and the remaining $n-k+1$ vertices. Equivalently, G_{star} is a star $K_{1,n-k+1}$ with its center vertex "blown up" to a $(k-1)$-clique.

As our main result, we find that if k is sufficiently small or sufficiently large (i.e. sufficiently close to $n/2$), then the k-matching-free process a.a.s. produces an Erdős-Gallai extremal graph, as follows.

Theorem 1. *If $k=o(n)$ then a.a.s. $G(N)\cong G_{\text{star}}$. This is tight; if $k=\Omega(n)$ then $G(N)\ncong G_{\text{star}}$ with probability $\Omega(1)$.*

Theorem 2. *If $k=n/2-o(\sqrt{n}/\log n)$ then a.a.s. $G(N)\cong G_{\text{clique}}$. This is essentially tight; if $k=n/2-\omega(\sqrt{n}/\log n)$then a.a.s. $G(N)\ncong G_{\text{clique}}$.*

The proofs of Theorem 1 and Theorem 2 are quite different to each other, and involve quite different methods to those typically used for studying H-free processes. In particular, we do not require the so-called *differential equation method.* The positive and negative parts of Theorem 1 will be proved separately in Section 2 and Section 3, and Theorem 2 will be proved in Section 4. We remark that while we made no particular attempt to consider the case $k=n/2-\Theta(\sqrt{n}/\log n)$, we expect that our proof of Theorem 2 can be modified to show that in this case $G(N)\ncong G_{\text{clique}}$ with probability $\Omega(1)$.

The regime where $k=\Omega(n)$ and $n-2k=\omega(\sqrt{n}/\log n)$ is significantly more challenging to study. As a first step, we show that if $k\le\varepsilon n$ for small ε, then a.a.s. $G(N)$ resembles G_{star}. Observe that G_{star} has independence number $n-k+1$ and $k-1$ vertices of degree $n-1$.

Proposition 3. *For all k, a.a.s. $G(N)$ has an independent set of size $n-(1+O(k/n))k$, and at least $(1-O(k/n))k$ vertices with degree $n-1$.*

The proof of Theorem 1 suggests that Proposition 3 is actually far from best possible; we suspect that if $k\le\varepsilon n$ for small ε, the error term $O(k/n)$ can be substantially improved. However, we observe that there is in fact a range of k in which $G(N)$ does not resemble any extremal k-matching-free graph. Observe that G_{clique} has $n-2k+1$ isolated vertices, and as before G_{star} has independence number $n-k+1$.

Proposition 4. *The following hold.*

(1) There is a constant $c<1/2$ such that if $k\ge cn$ then a.a.s. $G(N)$ has independence number $n-k-\Omega(n)$.

(2) If $k=\Omega(n)$ and $n-2k=\Omega(n)$ then a.a.s. $G(N)$ has $n-2k-\Omega(n)$ isolated vertices.

That is to say, there is a range of $\Theta(n)$ values of k for which the outcome of the k-matching-free process is typically substantially different from the Erdős-Gallai extremal graphs, in the sense that edges incident to an $\Omega(1)$-proportion of its vertices must be changed to arrive at either G_{star} or G_{clique}. We will give simple proofs of Proposition 3 and Proposition 4 in Section 5 and Section 6, respectively. Also, we remark that for Proposition 4, we can take $c=1/2-e^{-13}/2$, but no effort was made to optimise this constant.

Finally, recall that a vertex cover of a graph is a set of vertices such that every edge in the graph is incident to one of the vertices of this set. The problem of finding a maximum matching in a graph is in a certain sense dual to the problem of finding a minimum vertex cover, and the matching number (maximum size of a matching) and vertex cover number (minimum size of a vertex cover) are often considered together. Therefore one might naturally consider the restricted covering process $G^{\mathrm{vc}}(1), \dots, G^{\mathrm{vc}}(N)$, where we accept an edge $e(t)$ if and only if the vertex cover number would stay below k. However, in sharp contrast to the k-matching-free process, this restricted covering process exhibits quite trivial behaviour. One can easily check that, up to isomorphism, G_{star} is the only graph which is saturated with respect to the property of having vertex cover number less than k, so we will always have $G^{\mathrm{vc}}(N) \cong G_{\mathrm{star}}$.

1.1 Notation

For a probability distribution $\mathcal{L}$, we write $X \in \mathcal{L}$ to denote that a random element has distribution L. We write $\mathbb{G}(n, m)$ for the distribution of a uniformly random m-edge subset of K_n (this is known as the Erdős-Rényi random graph), and we use the same notation $\mathbb{G}(n, p)$ for the *binomial* random graph where each edge of K_n is present independently with probability p. Also, for $0 \le t \le N = \binom{n}{2}$, let $G^{\mathrm{all}}(t)$ be the graph with all the edges $e(1), \dots, e(t)$. This graph has precisely the Erdős-Rényi distribution $\mathbb{G}(n, t)$.

For a real number x, the floor and ceiling functions are denoted $\lfloor x \rfloor = \max\{i \in \mathbb{Z} : i \le x\}$ and $\lceil x \rceil = \min\{i \in \mathbb{Z} : i \ge x\}$. For a positive integer i, we write $[i]$ for the set $\{1, 2, \dots, i\}$. For real numbers x, y, we write $x \vee y$ to denote $\max\{x, y\}$ and we write $x \wedge y$ to denote $\min\{x, y\}$. All logs are base e.

Finally, we use standard asymptotic notation throughout, as follows. For functions $f = f(n)$ and $g = g(n)$ we write $f = O(g)$ to mean there is a constant C such that $|f| \le C|g|$, we write $f = \Omega(g)$ to mean there is a constant $c > 0$ such that $f \ge c|g|$, we write $f = \Theta(g)$ to mean that $f = O(g)$ and $f = \Omega(g)$, and we write $f = o(g)$ or $g = \omega(f)$ to mean that $f/g \to 0$. All asymptotics are taken as $n \to \infty$.

2 The positive part of Theorem 1

In this section we prove that if $k = o(n)$ then a.a.s. $G(N) \cong G_{\mathrm{star}}$. The proof consists of two phases. First, we track the unconstrained evolution of the process until we first see a matching of size $k-1$. During this time, the k-matching-free process is identical to the basic Erdős-Rényi random graph process, and is thus quite easy to analyse. In the second phase, we begin to track the formation of "augmenting paths" that would allow us to extend a $(k-1)$-matching into a k-matching, and are thus forbidden. To this end, we will define an evolving partition of the vertex set into "components" of vertices connected by certain special kinds of paths. We will then couple the k-matching-free process with a much simpler random graph process that captures this component structure, and study this simpler process via comparison with a certain binomial random graph.

2.1 The initial unconstrained evolution

Let $\nu(G)$ be the matching number of a graph G, and note that deterministically we have $\nu(G(t)) - \nu(G(t-1)) \in \{0, 1\}$. So, before the matching number reaches $k-1$, we accept every edge. Let $\tau = \min\{t : \nu(G(t)) = k-1\} \ge k-1$ be the time that the matching number reaches $k-1$. In this subsection we collect some simple a.a.s. properties of τ and $G(\tau)$.

Lemma 5. *A.a.s. $\tau \le 2k$.*

Proof. For $t \le 2k$, $G^{\mathrm{all}}(t-1)$ has at most $2k$ edges (comprising at most $4k$ vertices), so the probability $e(t)$ intersects these edges is at most $4kn/(\binom{n}{2} - 2k) \le 9k/n$. Therefore, the expected number of steps $t \le 2k$ which do not increase the matching number is at most $2k(9k/n) = 18k^2/n = o(k)$. By Markov's inequality, this number of steps is a.a.s. at most k, which proves that a.a.s. $\tau \le 2k$. $\square$

It follows from Lemma 5 that if we can prove that a decreasing property holds a.a.s. for the Erdős-Rényi random graph $G^{\mathrm{all}}(2k) \in \mathbb{G}(n, 2k)$, then it holds a.a.s. for $G(\tau)$. In fact, using say [24, Proposition 1.15], it suffices to show that such a property holds a.a.s. for the binomial random graph $\mathbb{G}(n, p)$, where $p = 2k/N$.

Lemma 6. *A.a.s. $G(\tau)$ is acyclic.*

Proof. We show that a.a.s. $G \in \mathbb{G}(n, p)$ is acyclic. Noting that $np = \Theta(k/n) = o(1)$, the expected number of cycles in G is

$$\sum_{i=3}^{n} \binom{n}{i} (i-1)! p^i \le \sum_{i=1}^{\infty} \frac{(np)^i}{i} = -\log(1 - np) = o(1),$$

and the desired result follows from Markov's inequality. $\square$

Next we show that most components of $G(\tau)$ are small. Define the *susceptibility* $S(G)$ of a graph G to be the sum of squares of sizes of its components. See for example [23] for background on this notion. Let $\tilde{S}(G)$ be $S(G)$ minus the number of isolated vertices of G (equivalently, $\tilde{S}(G)$ is the sum of squares of sizes of nontrivial components of G).

Lemma 7. *A.a.s. $\tilde{S}(G(\tau)) = o(n)$*

Proof. Let $G \in \mathbb{G}(n, p)$; we will show that a.a.s. $\tilde{S}(G) = o(n)$. Let X_v be the size of the component of v in G. Conditioning on the neighbourhood $N_G(v)$ of v in G, we have

$$X_v \le 1 + \sum_{w \in N_G(v)} X_w^v,$$

where $X_w^v \le X_w$ is the size of the component of w in $G - v$. Note that X_w^v does not actually depend on $N_G(v)$, so $\mathbb{E}[X_w^v \mid N_G(v)] = \mathbb{E}X_w^v \le \mathbb{E}X_w$ for all $w \ne v$. Then

$$\begin{aligned}
\mathbb{E}[X_v \mid N_G(v)] &\le 1 + \sum_{w \in N_G(v)} \mathbb{E}X_w, \\
\mathbb{E}X_v &\le 1 + (n-1)p\, \mathbb{E}X_v, \\
(1 - np)\mathbb{E}X_v &\le 1, \\
\mathbb{E}X_v &= 1 + o(1).
\end{aligned}$$

Let Q be the number of isolated vertices in G, so $\mathbb{E}Q = n(1-p)^{n-1} = ne^{O(np)} = n - o(n)$ and $\mathbb{E}\tilde{S}(G) = \mathbb{E}[\sum_v X_v] - \mathbb{E}Q = o(n)$. The desired result follows from Markov's inequality. $\square$

In view of the above lemmas, for the rest of the proof condition on an outcome of $\tau, e(1), \dots, e(\tau)$ such that $\tau \le 2k$, and such that $G(\tau)$ is acyclic and satisfies $\tilde{S}(G(\tau)) = o(n)$. Fix a $(k-1)$-edge matching M in $G(\tau)$, let A be its vertex set, and let $B = [n] \setminus A$ contain the other vertices. For any vertex $a \in A$, let m_a be the unique neighbour of a in M. Note that M will be a maximum matching in $G(t)$ for each $t \ge \tau$, by the definition of the process. Given our conditioning, note that $e(\tau + 1), \dots, e(N)$ is a uniformly random ordering of the pairs of vertices other than $e(1), \dots, e(\tau)$.

Now, Berge's Lemma [2, Theorem 1] says that a matching is maximum if and only if there is no *augmenting path*: that is, a path that starts and ends on unmatched vertices, and alternates between edges in and not in the matching. This means that each incoming edge $e(t)$ will be accepted if and only if its addition to $G(t-1)$ does not create an augmenting path with respect to M. For the rest of the paper, "augmenting path" will refer to a path that starts and ends in B, and alternates between edges in M and not in M. In order to keep track of the formation of such alternating paths, we introduce some auxiliary data ("charges" and "roots"), which evolve with $G(t)$, as follows.

2.2 Charges and roots

We will define charges $c_v(t) \in \{-1, 0, 1\}$ and roots $r_v(t) \in \{0\} \cup B$ for each $t \geq \tau$ and each vertex v. If the root of a vertex is zero we say it has no root, and if the charge of a vertex is zero we say it is uncharged. To begin with, only the vertices in B will be charged, and as the process $(G(t))_t$ evolves, the vertices in A will gradually become charged, gaining root data as this happens (charged vertices will never change their charge or root). The idea is that if a vertex is charged, that means there is an alternating path from that vertex to its root, and the sign of the charge corresponds to the parity of the length of this path. This information will allow us to deduce that certain edges are forbidden by the process.

First, we define "initial conditions", which do not actually correspond to charge and root data at any point of the process, but which will be used as a starting point to define the evolution of the charge and root data. For each $b \in B$, let $c_b(*) = -1$ and $r_b(*) = b$, meaning that each vertex in B has negative charge and has itself as a root. For each $a \in A$, let $c_a(*) = 0$ and $r_a(*) = 0$, meaning that each vertex in A has no charge and no root.

Next we describe how the data update at each step. For a graph G, and for charge and root data (c, r), define $c'(G, c, r)$ and $r'(G, c, r)$ via the following procedure. Start with the charges and roots given by c and r, and repeatedly do the following. As long as there is an edge in G between a negatively charged vertex v and an uncharged vertex $a \in A$, give a positive charge to a, give a negative charge to m_a, and give both of these newly charged vertices the same root as v. (If there are multiple edges between negatively charged and uncharged vertices, choose the one that was offered first).

Finally, we can define the charge and root data associated with each $G(t)$, $t \geq \tau$. Let $c(\tau) = c'(G(\tau), c(*), r(*))$ and $r(\tau) = r'(G(\tau), c(*), r(*))$, and for $t > \tau$ let $c(t) = c'(G(t), c(t-1), r(t-1)) = c'(G(t), c(*), r(*))$ and $r(t) = r'(G(t), c(t-1), r(t-1)) = r'(G(t), c(*), r(*))$. For $t \geq \tau$ and $b \neq 0$ let $C^b(t) = \{a \in A : r_a(t) = b\}$ be the "charge component" of vertices in A which have root b, and let $\mathcal{C}(t)$ be the collection of all such components which are nonempty. Note that the edges that were used to charge the vertices of $C^b(t)$ form a tree $T^b(t)$ on the vertex set $C^b(t) \cup \{b\}$, rooted at b. Also, let $\mathcal{D}(t)$ be the set of connected components in the subgraph of $G(t)$ induced by the $c(t)$-uncharged vertices, and define $\mathcal{F}(t) = \mathcal{C}(t) \cup \mathcal{D}(t)$ as the set of "generalised components", which partition A. See Figure 1 for an illustration.

We will next show that to prove Theorem 1 it suffices, roughly speaking, to prove that edges within generalised components are much rarer than edges between A and B. To state this as a lemma, we define some hitting times, for each $a \in A$. (Formally, we allow these hitting times to take the value ∞ if their corresponding events never occur).

- Let τ_a^F be the first time $t > \tau$ that we are offered an edge $e(t)$ between a and the rest of its generalised component in $\mathcal{F}(t-1)$, or between a and $r_a(t-1)$.
- Let τ_a^B be the first time $t > \tau$ that we are offered an edge $e(t)$ between a and $B \setminus \{r_a(t-1)\}$. Note that $B \setminus \{r_a(t-1)\} = B$ if a is uncharged at time $t-1$. Note also that $\tau_a^B < \infty$ because we are assuming that $\tau < 2k < |B|$.

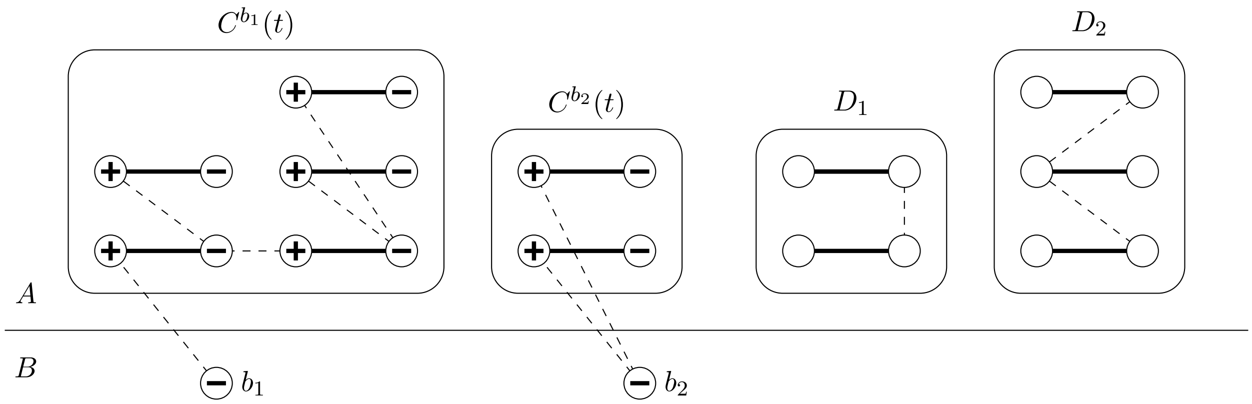

Figure 1. An example of the state of the charge and root data at some time $t \geq \tau$. The solid edges are edges of M, and $\mathcal{D}(t) = \{D_1, D_2\}$. Only the edges in the trees $T^b(t)$, and the edges in the uncharged components, are depicted.

- Let $\tau_a^C \leq \tau_a^B$ be the time $t \geq \tau$ at which a becomes charged.

Lemma 8. *If $\tau_v^F > \tau_{m_v}^B$ for all $v \in A$, then $G(N) \cong G_{\text{star}}$.*

Proof. We will show that if $\tau_v^F > \tau_{m_v}^B$ for all v, then $G(N)$ has no edges between negatively charged vertices, which implies that $G(N) \cong G_{\text{star}}$. Indeed, at the end of the process each edge of M will have one positively charged and one negatively charged vertex, so there will be $k-1$ negatively charged vertices in A. Combined with the $n - 2(k-1)$ negatively charged vertices in B, we will have proved that $G(N)$ has an independent set of size $n - k + 1$, which means it is isomorphic to a subgraph of G_{star}. But $G(N)$ is k-matching-saturated, so it cannot be a proper subgraph of the k-matching-free graph G_{star}.

Note first that there can never be any edge between negatively charged vertices with different roots $b, b' \in B$, because this would give an augmenting path between b and b'. Now, we consider the possible ways that an edge between negatively charged vertices with the same root could arise. The simplest possibility is that we could accept an edge $e(t)$ between two such vertices that are already negatively charged. The second possibility is that the process of charging vertices (via the introduction of an edge $e(t)$ between a negatively charged vertex v with root b and an uncharged vertex a) can somehow result in the previously uncharged endpoints of an existing edge e becoming negatively charged. Observe that this second possibility can only occur if e was previously in a cycle in its uncharged component. Indeed, the entire subtree T of $T^b(t)$ rooted at a would have been newly charged at step t, and since the charges $c(t)$ give a proper 2-colouring of T, $T \cup \{e\}$ must have had a cycle. Since we are assuming that $G(\tau)$ is acyclic, it suffices to prove:

(1) we never accept an edge that creates a cycle among the uncharged vertices, and;

(2) we never accept an edge between two negatively charged vertices with the same root.

First, since $\tau_v^F > \tau_{m_v}^B \geq \tau_v^C$ for each v, we are never even offered an edge $e(t)$ between an uncharged vertex and the rest of its component $D \in \mathcal{D}(t-1)$. This immediately proves (1).

Next, consider a vertex v which becomes negatively charged (with root b) at time τ_v^C. Let S_v be the set of negatively charged vertices $w \in C^b\big(\tau_v^C\big) \cup \{b\}$ such that the unique path between b and w in the tree $T^b\big(\tau_v^C\big)$ does not pass through v. (This set S_v does not evolve with t). Note that at any time t, for any negatively charged distinct $v, w \in C^b(t) \cup \{b\}$, we always have $w \in S_v$ or $v \in S_w$ (in particular, we will have $w \in S_v$ if v was charged later than w). The relevance of these sets is that if there is already an edge from m_v to $B \setminus \{b\}$, then an edge from v to S_v would create an augmenting path, so is forbidden.

For any $v \in A$, note that if conditions (1) and (2) have not been violated yet at time $\tau^B_{m_v} - 1$, then $G\big(\tau^B_{m_v} - 1\big)$ has an independent set of size $n - k + 1$, consisting of the negatively charged vertices and one colour class (not containing m_v) of a 2-colouring of the uncharged vertices. Unless m_v is already negatively charged, meaning that v is positively charged, this independent set would not be affected by the addition of $e\big(\tau^B_{m_v}\big)$, so $G(t-1) + e(t)$ has no k-matching and $e\big(\tau^B_{m_v}\big)$ is accepted. Since $\tau^B_{m_v} < \tau^F_v$, this means that if v is ever negatively charged then no edge between v and S_v can ever be accepted. Applying this argument iteratively to all v, in order of $\tau^B_{m_v}$, proves (2). □

Now, to prove that edges within generalised components are much rarer than edges between A and B, it will suffice to show that most generalised components are likely to remain "small" throughout the process. For a partition $\mathcal{G}$ of A, let $S(\mathcal{G})$ be the sum of squares of sizes of its parts. It would be most natural to try to show that $S(\mathcal{F}(t))$ is small for each t, but for technical reasons it is more convenient to individually deal with the $\mathcal{C}(t)$ and $\mathcal{D}(t)$. We can view each $\mathcal{C}(t)$ (respectively, each $\mathcal{D}(t)$) as a partition of A by putting each uncharged (respectively, charged) vertex in its own singleton part. Note that the sequence of partitions $\mathcal{C}(t)$ is "monotone" in the sense that for each $t > \tau$, $\mathcal{C}(t-1)$ is a refinement of $\mathcal{C}(t)$. This is not true for the $\mathcal{D}(t)$, because when part of an uncharged component gains charge, it splits into singleton components. So, let $\overline{\mathcal{D}}(t)$ be the finest common coarsening of the partitions $\mathcal{D}(\tau), \dots, \mathcal{D}(t)$. Equivalently, this means that $\overline{\mathcal{D}}(t)$ is the set of connected components of the union of all the uncharged subgraphs up to time t. The following lemma reduces Theorem 1 to a.a.s. bounds on $S(\mathcal{C}(N))$ and $S\big(\overline{\mathcal{D}}(N)\big)$.

Lemma 9. *To prove that a.a.s. $\tau^F_v > \tau^B_{m_v}$ for all $v \in A$, it suffices to prove that a.a.s.*

$$S(\mathcal{C}(N)),\ S(\overline{\mathcal{D}}(N)) = o(n).$$

Proof. For each $v \in A$, let $F_v(t) \in \mathcal{F}(t)$, $C_v(t) \in \mathcal{C}(t)$ and $D_v(t) \in \overline{\mathcal{D}}(t)$ be the parts containing v in the partitions $\mathcal{F}(t)$, $\mathcal{C}(t)$ and $\overline{\mathcal{D}}(t)$ respectively. Let $X_v(t) = |C_v(t)| + |D_v(t)|$, and let $S^X(t) = \sum_{v \in A} X_v(t)$. So, $|F_v(t)| \le X_v(t)$, the sequence of $S^X(t)$ is monotone nondecreasing in t, and we are assuming that a.a.s. $S^X(N) = S(\mathcal{C}(N)) + S(\overline{\mathcal{D}}(N)) = o(n)$.

By considering the events $\{t = \tau^F_v < \tau^B_{m_v}\}$ for each $t \ge \tau$ and $v \in A$, it would be possible to show that

$$\Pr\left(\bigcup_{v \in A} \{\tau^F_v < \tau^B_{m_v}\}\right) \le \frac{2 \sum_{v \in A} \mathbb{E} X_v(N)}{n} = \frac{2\,\mathbb{E} S^X(N)}{n}.$$

However, an a.a.s. bound on $S^X(N)$ does not (directly) imply that $\mathbb{E} S^X(N) = o(n)$. To overcome this difficulty, we essentially stop the process as soon as $S^X(N)$ gets too large. To be precise, choose $f = o(n)$ such that a.a.s. $S^X(N) \le f$, and let

$$\tau^S_f = \min\big\{t \le N : S^X(t) > f\big\} \wedge (N+1),$$

so that a.a.s. $\tau^S_f = N + 1$, and therefore a.a.s. $\tau^B_{m_v} \wedge \tau^S_f = \tau^B_{m_v}$. Then, $S^X(\tau^S_f - 1) \le f = o(n)$, so it suffices to show that

$$\Pr\left(\bigcup_{v \in A} \{\tau^F_v \le \tau^B_{m_v} \wedge \tau^S_f\}\right) \le \frac{2\,\mathbb{E} S^X(\tau^S_f - 1)}{n}. \tag{1}$$

Consider any $t > \tau$ and $v \in A$, and condition on $e(\tau + 1), \dots, e(t-1)$. If $t - 1 < \tau^F_v \wedge \tau^B_{m_v} \wedge \tau^S_f$ then there are at most $|(F_v(t-1) \cup \{r_v(t-1)\}) \setminus \{v\}| \le |F_v(t-1)|$ choices for $e(t)$ that would cause

$t = \tau_v^F$. There are at least $|B \setminus \{r_v(t-1)\}| - \tau \geq n/2$ choices for $e(t)$ that would cause $t = \tau_v^F \wedge \tau_{m_v}^B$. (Here we account for the fact that τ edges had already been offered at time τ, and are therefore not viable candidates for $e(t)$). So,

$$\begin{aligned}
&\Pr\bigl(\tau_v^F = t \leq \tau_f^S \mid \tau_v^F \wedge \tau_{m_v}^B = t,\, e(\tau+1), \ldots, e(t-1)\bigr) \\
&\quad = \frac{\Pr\Bigl(\tau_v^F = t \leq \tau_f^S \Bigm| e(\tau+1), \ldots, e(t-1)\Bigr)}{\Pr\Bigl(\tau_v^F \wedge \tau_{m_v} = t \Bigm| e(\tau+1), \ldots, e(t-1)\Bigr)} \\
&\quad \leq \frac{|F_v(t-1)|}{n/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Pr\bigl(\tau_v^F \leq \tau_{m_v}^B \wedge \tau_f^S \mid \tau_v^F \wedge \tau_{m_v}^B = t\bigr) &= \Pr\bigl(\tau_v^F = t \leq \tau_f^S \mid \tau_v^F \wedge \tau_{m_v}^B = t\bigr) \\
&\leq \frac{2\,\mathbb{E}\Bigl|F_v\Bigl(\tau_f^S - 1\Bigr)\Bigr|}{n} \leq \frac{2\,\mathbb{E}X_v(\tau_f^S - 1)}{n}.
\end{aligned}$$

Since this holds for all t, we in fact have

$$\Pr\bigl(\tau_v^F \leq \tau_{m_v}^B \wedge \tau_f^S\bigr) \leq \frac{2\,\mathbb{E}X_v(\tau_f^S - 1)}{n}.$$

The desired inequality (1) follows, by the union bound. $\square$

2.3 Coupling with a simpler process

In this section we define an auxiliary random graph process $G'(t)$ based on $G(t)$, which captures most of its generalised component structure but is much simpler to analyse. Each $G'(t)$ will be a graph on the vertex set A. To start with, define $G'(\tau)$ to contain a clique on the vertex set of each generalised component $F \in \mathcal{F}(\tau)$. Then, for every $t > \tau$ and every $v \in A$, let $E_v(t)$ be the event that there was an edge $e(t')$, with $\tau \leq t' \leq t$, between B and v. Let $e(t) = \{v(t), w(t)\}$, and for all $t > \tau$, let

$$G'(t) = \begin{cases} G'(t-1) & \text{if } E_{v(t)}(t) \text{ and } E_{w(t)}(t) \text{ both hold, or if } e(t) \not\subseteq A; \\ G'(t-1) + e(t) & \text{otherwise.} \end{cases}$$

That is to say, we reject an edge within A only if both of its endpoints have already been offered an edge to B.

Now, let $\mathcal{F}'(t)$ be the set of connected components of $G'(t)$. We would like to be able to say that for each t, the partition $\mathcal{F}(t)$ is a refinement of the partition $\mathcal{F}'(t)$, so that we can control $S(\mathcal{C}(N))$ and $S(\overline{\mathcal{D}}(N))$ via $S(\mathcal{F}'(N))$. This turns out to be almost true, except for the fact that an uncharged component D can merge with a charge component $C^b(t-1)$ via an edge $e(t)$ from D to b, and such edges are "invisible" to the process $G'(t)$.

So, we define a slight refinement $\mathcal{C}'(t)$ of $\mathcal{C}(t)$ ignoring these "invisible" edges, such that $\mathcal{C}'(t)$ really is a refinement of $\mathcal{F}'(t)$; we will bound $S(\mathcal{C}'(N))$ via $S(\mathcal{F}'(N))$ and deal with the coarsening of $\mathcal{C}'(t)$ to $\mathcal{C}(t)$ separately. First we define the refinement $\mathcal{C}_{\text{sub}}(t)$ of $\mathcal{C}(t)$ obtained by splitting each $C^b(t)$ into the connected components of the forest $T^b(t) - b$ (we call these "sub-components"). Then let $\mathcal{C}'(t)$ be the finest common coarsening of $\mathcal{C}_{\text{sub}}(t)$ and $\mathcal{C}(\tau)$. That is to say, we only group vertices which are in the same charge component because they were directly charged by each other, not by their external root, unless they had already been charged in this way by time τ.

Lemma 10. *$\mathcal{C}'(N)$ and $\overline{\mathcal{D}}(N)$ are both refinements of $\mathcal{F}'(N)$, as partitions of A.*

Proof. Let $\mathcal{G}(t) = \mathcal{D}(t) \cup \mathcal{C}'(t)$. Since $\mathcal{F}'(t)$ is monotone in the sense that each $\mathcal{F}'(t-1)$ is a refinement of $\mathcal{F}(t)$, it suffices to prove that for each $\tau \le t \le N$, $\mathcal{G}(t)$ is a refinement of $\mathcal{F}'(t)$. First note that $\mathcal{G}(\tau) = \mathcal{F}'(\tau)$ by definition. Now, there are two ways $\mathcal{G}(t)$ can differ from $\mathcal{G}(t-1)$. The first possibility is that an edge $e(t)$ is accepted for $G(t)$ between two uncharged vertices in different components $D_1, D_2 \in \mathcal{D}(t-1)$, in which case those components are merged to give $\mathcal{D}(t)$. The second possibility is that $e(t)$ is accepted for $G(t)$ between an uncharged vertex (in a component $D \in \mathcal{D}(t)$, say) and a negatively charged vertex w (with root b, say), in which case some subset $U \subseteq D$ gains charge and is added to the relevant sub-component of $C^b(t-1)$. In this case, w could be b itself (in which case $\mathcal{G}(t)$ is a refinement of $\mathcal{G}(t-1)$), or w could be in $C^b(t-1) \subseteq A$. Considering all possibilities, it suffices to show that every edge between an uncharged vertex and another vertex in A, which is accepted for $G(t)$, is also accepted for $G'(t)$. To see this, note that if $E_v(t)$ holds, then $t \ge \tau_v^B \ge \tau_v^C$, meaning that v is charged at time t. □

Now we show how to control $S(\mathcal{C}(N))$ and $S\big(\overline{\mathcal{D}}(N)\big)$ via $S(\mathcal{F}'(N)) = S(G'(N))$.

Lemma 11. *To prove that a.a.s. $S(\mathcal{C}(N)), S\big(\overline{\mathcal{D}}(N)\big) = o(n)$, it suffices to prove that a.a.s. $S(G'(N)) = o(n)$.*

Proof. First note that by Lemma 10, a.a.s. $S(\overline{\mathcal{D}}(N)) \le S(\mathcal{F}'(N)) = o(n)$.

Next, let $\mathcal{E}$ be the relative ordering of all edges in the sequence $e(\tau+1), \dots, e(N)$ which are between A and B. Recalling that we are conditioning on $e(1), \dots, e(\tau)$, and observing that $\mathcal{C}'(t)$ can only differ from $\mathcal{C}'(t-1)$ if $e(t)$ is between two vertices in A, note that $\mathcal{C}'(N)$ does not depend on $\mathcal{E}$. By Lemma 10, we can assume that a.a.s. $S(\mathcal{C}'(N)) \le S(\mathcal{F}'(N)) = o(n)$, so condition on an outcome of $\mathcal{C}'(N)$ with this property. This does not change the distribution of $\mathcal{E}$.

Now, for $C \in \mathcal{C}'(N)$, let $r(C)$ be the common root of all elements of C, corresponding to the first edge that was offered between C and B. For some $C \in \mathcal{C}'(N)$, we may have already seen an edge between C and B by time τ, meaning that $r(C)$ is determined. Let $\mathcal{Q}$ be the set of such C. For all other C, by the randomness of $\mathcal{E}$, the root $r(C)$ is uniformly distributed in B, and these roots are independent of each other.

For each $C \in \mathcal{C}'(N)$, let $C^* \in \mathcal{C}(N)$ be charge component which includes C. We will estimate each $\mathbb{E}\,|C^*|$. To this end, note that

$$C^* = \bigcup_{\substack{C' \in \mathcal{C}'(N): \\ r(C') = r(C)}} C'.$$

Now, in in the case where $C \notin \mathcal{Q}$, for each $C' \ne C$ we have $r(C) = r(C')$ with probability $1/|B| = (1+o(1))/n$, so

$$\mathbb{E}\,|C^*| \le |C| + (1+o(1)) \sum_{C' \in \mathcal{C}'(N)} \frac{|C'|}{n} = |C| + o(1).$$

Alternatively, if $C \in \mathcal{Q}$ then C is the only component in $\mathcal{Q}$ with root $r(C)$, and for each $C' \notin \mathcal{Q}$ we have $r(C) = r(C')$ with probability $1/|B|$, so

$$\mathbb{E}\,|C^*| \le |C| + (1+o(1)) \sum_{C' \in \mathcal{C}'(N) \setminus \mathcal{Q}} \frac{|C'|}{n} = |C| + o(1).$$

Combining both cases, with $C_v \in \mathcal{C}'(N)$ as the part containing v in $\mathcal{C}'(N)$, we have

$$\mathbb{E}\, S(\mathcal{C}(N)) = \sum_{v \in A} \mathbb{E}\,|C_v^*| \le \sum_{v \in A} |C_v| + o(|A|) = O\big(S(\mathcal{C}'(N))\big) = o(n).$$

The desired result follows by Markov's inequality. ☐

Now, apart from the edges in $G'(\tau)$, note that each edge in $G'(N)$ is present with probability $\Theta(1/n)$. This is because the condition for including an edge $\{u, v\}$ is that it must be offered before any of the edges between u and B, or before any of the edges between v and B, and there are $\Theta(n)$ edges of both types. Although the edges of $G'(N)$ are not independent, we will see in the next subsection that we can nevertheless reduce our problem to a comparable problem concerning a certain binomial random graph where each edge is independently present with probability $1/n$. We remark that the susceptibility of the union of a random graph and a fixed sparse random-like graph has already been studied, by Spencer and Wormald [33] and by Bohman, Frieze, Krivelevich, Loh and Sudakov [4]. However our methods will be much simpler, and will resemble the proof of Lemma 7.

2.4 Reducing to a binomial random graph

First we define intermediate random graphs G^{Unif}, G^{Exp} on the vertex set A, which stochastically dominate $G'(N)$. Let $B' \subseteq B$ be a set of $n - o(n)$ isolated vertices in $G(\tau)$, let $(\eta_{a,b})_{a\in A,\, b\in B'}$ and $(\eta_e)_{e\in\binom{A}{2}}$ be independent random variables uniformly distributed in the interval $[0, 1]$. Almost surely each is distinct, so these random variables induce a uniformly random ordering of the possible edges within A and between A and B. Put an edge $e = \{v, w\}$ in G^{Unif} if $e \in G'(\tau)$, or if $\eta_e \le (\min_{b\in B'} \eta_{v,b}) \vee (\min_{b\in B'} \eta_{w,b})$. This graph is defined so that it stochastically dominates $G'(N)$.

Now, note that the uniform distribution $\mathrm{Unif}(0, 1)$ is stochastically dominated by the exponential distribution $\mathrm{Exp}(1)$ (one can see this by comparing cumulative distribution functions), and recall that the minimum of m independent $\mathrm{Exp}(1)$ random variables has the distribution $\mathrm{Exp}(m)$. So, let $(\gamma_v)_{v\in A}$ be independent $\mathrm{Exp}(|B'|)$ random variables, and define G^{Exp} by putting an edge $e = \{v, w\}$ in G^{Exp} if $e \in G'(\tau)$, or if $\eta_e \le \gamma_v \vee \gamma_w$. Then G^{Exp} stochastically dominates G^{Unif}.

Next, let G^* be the graph obtained by starting with $G'(\tau)$ and blowing up each vertex v into a clique of size $2\lfloor\gamma_v n\rfloor + 3$. This means that each vertex is replaced with a clique, and two vertices of G^* in different cliques are adjacent if their corresponding vertices in $G'(\tau)$ were adjacent. Given G^* (with vertex set V^*, say), let G^p be a random graph on the same vertex set V^*, where each of the $\binom{|V^*|}{2}$ edges is independently present with probability $1/n$. We will show that in a certain sense $G^* \cup G^p$ dominates G^{Exp}.

Lemma 12. *Let E be the event that each $\gamma_v \le 2\log n/n$, and condition on $(\gamma_v)_{v\in A}$ satisfying E. For each $v, w \in A$ with $v \ne w$, let $p_{v,w}$ be the probability that there is at least one edge between the blowup of v and the blowup of w, in G. Then $p_{v,w} \ge \gamma_v \vee \gamma_w$.*

Proof. We have

$$p_{v,w} := 1 - (1 - 1/n)^{(2\lfloor\gamma_v n\rfloor+3)(2\lfloor\gamma_w n\rfloor+3)}.$$

By Taylor's theorem (expanding around $p = 0$), for all $p, x \ge 0$,

$$(1 - p)^x \le 1 - px + p^2 x(x - 1)/2 \le 1 - px(1 - px),$$

and since each $\gamma_v \le 2\log n/n$, we have $(2\lfloor\gamma_v n\rfloor + 3)(2\lfloor\gamma_w n\rfloor + 3) \le 17\log^2 n$. So,

$$\begin{aligned} p_{v,w} &\ge \left(1 - 17\log^2 n/n\right)(2\lfloor\gamma_v n\rfloor + 3)(2\lfloor\gamma_w n\rfloor + 3)/n \\ &> \frac{1}{2}(2\gamma_v n + 1)(2\gamma_w n + 1)/n \\ &\ge \gamma_v + \gamma_w \ge \gamma_v \vee \gamma_w, \end{aligned}$$

as desired. ☐

Lemma 12 shows that G^{Exp} and $G^*\cup G^p$ can be coupled in such a way that $S(G^*\cup G^p)\geq S\big(G^{\mathrm{Exp}}\big)$ whenever E holds. Note that $\Pr(\gamma_v > x) = e^{-|B'|x}$, so $\Pr(\gamma_v > 2\log n/n) = o(1/n)$, and in fact E a.a.s. holds. Recalling that G^{Exp} stochastically dominates $G'(N)$, to prove Theorem 1 via Lemmas 8, 9 and 11 it suffices to prove the following lemma.

Lemma 13. *A.a.s.* $S(G^*\cup G^p) = o(n)$.

Proof. For each connected component $F\in\mathcal{F}'(\tau)$, let $X_F = \sum_{v\in F}\gamma_v$. The size of the corresponding blown-up component in G^* is $3|F|+2n\sum_{v\in F}\lfloor\gamma_v\rfloor \leq 3|F|+2nX_F$. Now, $\mathbb{E}\gamma_v = 1/|B'|$ and $\operatorname{Var}\gamma_v = 1/|B'|^2$, so $\mathbb{E}X_F = |F|/|B'|$, $\operatorname{Var}X_F = |F|/|B'|^2$ and $\mathbb{E}X_F^2 = |F|(|F|+1)/|B'|^2$. Recalling that $|B'| = n-o(n)$, we have

$$\mathbb{E}(3|F|+2nX_F)^2 = 9|F|^2+12n|F|\mathbb{E}X_F+4n^2\mathbb{E}X_F^2 = O\Big(|F|^2\Big).$$

The components of $G'(\tau)$ are subsets of nontrivial components of $G(\tau)$, so $S(G'(\tau))\leq\tilde{S}(G(\tau)) = o(n)$, and

$$\mathbb{E}S(G^*) = \sum_{F\in\mathcal{F}'(\tau)}\mathbb{E}(3|F|+2nX_F)^2 = O\left(\sum_{F\in\mathcal{F}'(\tau)}|F|^2\right) = O\big(S\big(G'(\tau)\big)\big) = o(n).$$

By Markov's inequality, a.a.s. $S(G^*) = o(n)$, so for the rest of the proof we condition on an outcome of G^* satisfying this property.

Now we proceed in a similar way to the proof of Lemma 7. Let F_v be the component of v in G^*, let G_v^* be obtained from G^* by deleting all edges incident to v, and let Y_v be the size of the component of v in $G_v^*\cup G^p$. Let $Y_w^v\leq Y_w$ be the size of the component of w in $(G_w^*\cup G^p)-v$. Conditioning on $N_{G^p}(v)$, we have

$$Y_v\leq 1+\sum_{u\in N_{G^p}(v)}\sum_{w\in F_u}Y_w^v.$$

Note that Y_w^v does not depend on $N_{G^p}(v)$, so $\mathbb{E}[Y_w^v\mid N_{G^p}(v)] = \mathbb{E}Y_w^v\leq\mathbb{E}Y_w$ for all $w\neq v$. If v is chosen to maximise $\mathbb{E}Y_v$, we have

$$\begin{aligned}
\mathbb{E}[Y_v\mid N_{G^p}(v)] &\leq 1+\sum_{u\in N_{G^p}(v)}\sum_{w\in F_u}\mathbb{E}Y_w\\
&\leq 1+\mathbb{E}Y_v\sum_{u\in N_{G^p}(v)}|F_u|,\\
\mathbb{E}Y_v &\leq 1+(S(G^*)/n)\,\mathbb{E}Y_v,\\
(1-o(1))\,\mathbb{E}Y_v &\leq 1,\\
\mathbb{E}Y_v &= 1+o(1).
\end{aligned}$$

Then, observe that the size of the component of w in $G^*\cup G^p$ is at most $\sum_{u\in F_w}Y_u$, so its expected size is $(1+o(1))|F_w|$ and

$$\mathbb{E}S(G^*\cup G^p) = (1+o(1))S(G^*) = o(n).$$

The desired result follows by Markov's inequality. $\square$

3 The negative part of Theorem 1

In this section we prove that, for any fixed $\varepsilon > 0$, if $\varepsilon n \le k \le (1-\varepsilon)n/2$, then $G(N) \not\cong G_{\mathrm{star}}$ with probability $\Omega(1)$. This suffices, because the case where $n - 2k = o(n)$ is handled by Proposition 4. Define τ as in Section 2.

Lemma 14. *There is $R = R(\varepsilon)$ such that a.a.s. $\tau \le Rn$.*

Proof. We prove that in the Erdős-Rényi random graph $G^{\mathrm{all}}(Rn) \in \mathbb{G}(n, Rn)$ there is a.a.s. a k-matching. Using [24, Proposition 1.15], it actually suffices to show this for a binomial random graph $G \in \mathbb{G}(n, R/n)$. To do this, we prove that there is no independent set of size εn. Indeed, for any set of εn vertices, the probability that the set is independent is

$$(1 - R/n)^{\binom{\varepsilon n}{2}} \le e^{-\Omega\left(\varepsilon^2 Rn\right)}.$$

If R is much larger than ε^{-2}, then this probability is $o(2^{-n})$, so the union bound says that a.a.s. G has no independent set of size εn, as desired. □

Now, say a triangle in a graph is *isolated* if there are no edges between the triangle and the rest of the graph.

Lemma 15. *$G(\tau)$ has an isolated triangle with probability $\Omega(1)$.*

Proof. Let R be as in Lemma 14, and consider independent binomial random graphs $G_1 \in \mathbb{G}(n, p_1)$ and $G_2 \in \mathbb{G}(n, p_2)$, with $p_1 = \varepsilon/n$ and $p_2 = 3R/n$. For any m, the distribution of G_1 (respectively $G_1 \cup G_2$) conditioned on the event that $e(G_1) = m$ (respectively, that $e(G_1 \cup G_2) = m$), is precisely the Erdős-Rényi distribution $\mathbb{G}(n, m)$. Note that a.a.s. G_1 has fewer than $k - 1 \le \tau$ edges, and a.a.s. G_2 has at least Rn edges. So, G_1, G_2 and $(G(t))_t$ can be coupled together such that a.a.s. $G_1 = G^{\mathrm{all}}(e(G_1)) \subseteq G(\tau)$ and $G_1 \cup G_2 = G^{\mathrm{all}}(e(G_1 \cup G_2)) \supseteq G(\tau)$. We will prove that G_1 has an isolated triangle with probability $\Omega(1)$, and we will then show that conditioning on an outcome of G_1 with an isolated triangle, that triangle remains isolated in $G_1 \cup G_2$ with probability $\Omega(1)$.

Let X be the number of isolated triangles in G_1, so that

$$\mathbb{E}X = \binom{n}{3} p_1^3 (1-p_1)^{3(n-3)} = (1 + o(1)) \frac{\varepsilon^3}{6} e^{-3\varepsilon} = \Theta(1),$$

and similarly

$$\mathbb{E}[X(X-1)] = \binom{n}{3}\binom{n-3}{3} p_1^6 (1-p_1)^{6(n-3)+9} = \Theta(1),$$
$$\mathbb{E}X^2 = \mathbb{E}[X(X-1)] + \mathbb{E}X = \Theta(1).$$

So, $(\mathbb{E}X)^2/\mathbb{E}X^2 = \Omega(1)$, and by the second moment method, G_1 has an isolated triangle with probability $\Omega(1)$. Now, condition on an outcome of G_1 with an isolated triangle T. The probability T is still isolated in $G_1 \cup G_2$ is $(1 - p_2)^{3n-9} = \Omega(1)$. □

Now, if $G(\tau)$ has an isolated triangle, then $G(N) \not\cong G_{\mathrm{star}}$. Indeed, fix an isolated triangle T and a maximum $(k-1)$-matching M in $G(\tau)$. Observe that exactly one edge of T is used in M. We can never accept an edge between T and the vertices not involved in M, because this would create an augmenting path. So, in $G(N)$ the vertices of T have degree less than $n - 1$, while in G_{star} there is no edge between vertices of degree less than $n - 1$. Therefore $G(N) \not\cong G_{\mathrm{star}}$, as desired.

4 Proof of Theorem 2

Let $f = n - 2k$. In this section, we define a new hitting time:

$$\tau = \min\{t : G(t) \text{ has exactly } f+1 \text{ isolated vertices}\}.$$

First we show that a.a.s. τ actually occurs.

Lemma 16. *If $f = o(n)$ then a.a.s. $\tau < \infty$. Moreover, a.a.s. for $t \le \tau$, each $e(t)$ is accepted.*

Proof. Let

$$\tau' = \min\{t : G(t) \text{ has at most } f+2 \text{ isolated vertices}\}.$$

For $t \le \tau'$, each $G(t-1)$ has more than $f+2$ isolated vertices, so has no matching of size $k-1$, meaning that every edge $e(t)$ is accepted (so $\tau' < \infty$). Adding an edge to a graph can destroy at most two isolated vertices, so $G(\tau')$ has $f+1$ or $f+2$ isolated vertices. Now, condition on any outcome for $G(\tau')$ with $f+2$ isolated vertices, and let W be the set of these isolated vertices. There are only $O\big(f^2\big)$ pairs of isolated vertices, but there are $(f+2)(n-(f+2)) = \Omega(nf) = \omega\big(f^2\big)$ choices of an isolated and non-isolated vertex. Therefore we will a.a.s. be offered an edge between W and $V \setminus W$ before we are ever offered an edge between isolated vertices. This edge (and all edges preceding it) will be accepted, resulting in a graph with $f+1$ isolated vertices. □

We now compute the approximate value of τ so that we may compare $G(\tau)$ to standard Erdős-Rényi/binomial random graphs. For $h > 0$ define

$$t_h^- = \left\lfloor (\log n - \log(f+1) - h)\frac{n}{2} \right\rfloor, \quad t_h^+ = \left\lceil (\log n - \log(f+1) + h)\frac{n}{2} \right\rceil.$$

Lemma 17. *Suppose $f = o(n)$. Then for any $h \to \infty$, a.a.s. $t_h^- \le \tau \le t_h^+$.*

Proof. We can assume h is sufficiently slowly-growing so that $t_h^- = \omega(n)$. Now, let X_p be the number of isolated vertices in $\mathbb{G}(n,p)$. If $np \to \infty$ and $np = O(\log n)$ then

$$\begin{aligned}
\mathbb{E}X_p &= n(1-p)^{n-1} \\
&= n\big(e^{-p} + O\big(p^2\big)\big)^{n-1} = ne^{-pn}\big(1 + O\big(p^2\big)\big)^n \\
&= ne^{-pn} + O\big(n^2p^2e^{-pn}\big) = (1+o(1))ne^{-pn}.
\end{aligned}$$

Note that $npe^{-np} = o(1)$, so a similar calculation gives

$$\mathbb{E}[X_p(X_p-1)] = n(n-1)(1-p)^{2n-3} \le n^2e^{-2pn} + O(n^3p^2e^{-2pn}) = (ne^{-pn})^2 + o(ne^{-pn}).$$

It follows that

$$\operatorname{Var} X_p = \mathbb{E}[X_p(X_p-1)] + \mathbb{E}X_p - (\mathbb{E}X_p)^2 = o\big(n^2e^{-2pn}\big) = o\big(\mathbb{E}X_p^2\big).$$

If $p \le (\log n - \log(f+1) - h)/n$ then $\mathbb{E}X_p = \omega(1)$ so a.a.s. $X_p \ge \mathbb{E}X_p/2 \ge f+2$ by Chebyshev's inequality. If $p \ge (\log n - \log(f+1) + h)/n$ then $\mathbb{E}X_p = O((f+1)e^{-h}) = o(f+1)$ so a.a.s. $X_p \le f+1$ by Markov's inequality. Using say [24, Proposition 1.15], in $G^{\mathrm{all}}(t_h^-)$ there are a.a.s. at least $f+2$ isolated vertices, and in $G^{\mathrm{all}}(t_h^+)$ there are a.a.s. at most $f+1$ isolated vertices. The first of these facts immediately implies $t_h^- < \tau$. Recalling from Lemma 16 that a.a.s. every edge up to time τ is accepted, the second of these facts implies that a.a.s. $t_h^+ \ge \tau$. □

Now, given $G(\tau)$, we will have $G(N) \cong G_{\mathrm{clique}}$ if and only if we reject all further edges involving an isolated vertex of $G(\tau)$. The proofs of the positive and negative parts of Theorem 2 now diverge.

4.1 Matching-resilience: the positive part

In this section we explain how to prove that if $f = o(\sqrt{n}/\log n)$ then after time τ we a.a.s. reject all edges involving isolated vertices.

Say a vertex in a graph is *dangerous* if it has a neighbour of degree 1, or if it has a neighbour of degree 2 and that neighbour is within distance 2 of another vertex of degree at most 2. Say a graph with an odd number of vertices is *matching-resilient* if after deleting any non-dangerous vertex there is a perfect matching. Let W be the set of isolated vertices in $G(\tau)$. The following lemma is crucial.

Lemma 18. *If $f = o(\sqrt{n}/\log n)$, then a.a.s. $G(\tau) \setminus W$ is matching-resilient.*

The proof of Lemma 18 is a bit involved, so we defer it to Section 4.3. Next, we also need a bound on the number of dangerous vertices. The slightly cumbersome definition of a dangerous vertex was carefully chosen so that the following lemma would hold.

Lemma 19. *If $f = o(\sqrt{n}/\log n)$, then a.a.s. $G(\tau)$ has $o(\sqrt{n})$ dangerous vertices.*

Proof. First we introduce some convenient terminology. By "1-vertex" we mean a vertex with degree at most 1, by "2-vertex" we mean a vertex with degree at most 2, and by "2-pair" we mean a pair of 2-vertices whose distance is at most 2. Every dangerous vertex is adjacent to a 1-vertex or adjacent to one of the vertices of a 2-pair, so it suffices to show that in $G(\tau)$ there are $o(\sqrt{n})$ 1-vertices and $o(\sqrt{n})$ 2-pairs.

Now, let $g = \sqrt{n}/(f \log n) \to \infty$ and choose $h \to \infty$ with $h = o(\log g)$. We consider the binomial random graph $G \in \mathbb{G}(n,p)$, with

$$p = \frac{\log n - \log(f+1) - h}{n} = \frac{\log n/2 + \log\log n + (1+o(1))\log g}{n}.$$

The expected number of 1-vertices in G is

$$n(1-p)^{n-1} + n(n-1)p(1-p)^{n-2} = o\big(\sqrt{n}\big),$$

and the expected number of 2-vertices is

$$o\big(\sqrt{n}\big) + O\big(n^3p^2(1-p)^n\big) = o\big(\sqrt{n}\log n\big).$$

Considering all possible cases for the structure of a 2-pair, the expected number of 2-pairs is

$$O\Big(n^2(1-p)^{2n}\big(p + np^2 + n^2p^3 + n^3p^4\big)\Big) = o\big(\log^2 n\big).$$

Let m be the number of edges of G; conditioned on m, G has the Erdős-Rényi distribution $\mathbb{G}(n,m)$, so we can couple G and $\big(G^{\mathrm{all}}(t)\big)_t$ in such a way that $G^{\mathrm{all}}(m) = G$. By the Chernoff bound, a.a.s.

$$\left| m - (\log n - \log(f+1) - h)\frac{n}{2} \right| = o(n),$$

meaning that $m < t^-_{h/2}$. So condition on an outcome of $G = G^{\mathrm{all}}(m)$ satisfying this property, such that G has at most $\sqrt{n}$ 1-vertices, at most $\sqrt{n}\log n$ 2-vertices, and at most $\log^2 n$ 2-pairs. Let $U(t)$ be the set of 2-vertices and their neighbours in $G^{\mathrm{all}}(t)$. Note that $G^{\mathrm{all}}(t)$ has at most $\sqrt{n}$ 1-vertices, and $|U(t)| \le 3\sqrt{n}\log n$, for all $t \ge m$. Now, after time m, the only way a new 2-pair can be formed is if we are offered an edge $e(t)$ between two vertices in $U(t-1)$. Note that $t^+_{h/2} - m = O(hn)$, so the expected number of such edges we will be offered before time $t^+_{h/2}$ is

$$O\left(\frac{\big(\sqrt{n}\log n\big)^2 hn}{n^2}\right) = O\big(h\log^2 n\big) = o\big(\log^3 n\big).$$

So, a.a.s. there are at most $\log^3 n$ times $m < t \le t^-_{h/2}$ in which new 2-pairs are created. Now, we will soon see in Lemma 24 that a.a.s. in each of $G^{\text{all}}\left(t^-_{h/2}\right), G^{\text{all}}\left(t^-_{h/2}+1\right), \dots, G^{\text{all}}\left(t^+_{h/2}\right)$ there are no three 2-vertices within distance 20 of each other, which means that only one new 2-pair can be created at a time. It follows that a.a.s. each of $G^{\text{all}}\left(t^-_{h/2}\right), G^{\text{all}}\left(t^-_{h/2}+1\right), \dots, G^{\text{all}}\left(t^+_{h/2}\right)$ have $o(\sqrt{n})$ 1-vertices and $O\left(\log^3 n\right)$ 2-pairs. By Lemma 17, a.a.s. $t^-_{h/2} \le \tau \le t^+_{h/2}$, implying that a.a.s. $G(\tau)$ has $o(\sqrt{n})$ dangerous vertices, as desired. □

In view of the above lemmas, condition on an outcome of $\tau, e(1), \dots, e(\tau)$ such that $\tau \le n \log n$, and $G(\tau)$ is matching-resilient, and $G(\tau)$ has $o(\sqrt{n})$ dangerous vertices. Let W and U be the sets of isolated and dangerous vertices of $G(\tau)$, respectively, let $V = [n]$ be the set of all vertices, and let $r = 2n \log n$. The probability we are offered an edge between W and U before time $\tau + r$ is

$$o\left(\frac{r(f+1)\sqrt{n}}{n^2}\right) = o(1).$$

It follows that a.a.s. $G(\tau + r)$ still has $f + 1$ isolated vertices. Indeed, any edge $e(t)$ between W and $V \setminus U$ is rejected, beacause by matching-resilience $G(\tau)$ has a $(k-1)$-matching not involving the vertices of $e(t)$. Next, the expected number of edges we are offered before time $\tau + r$ that involve W at all is $O(rf/n) = o(r)$. So, by Markov's inequality, a.a.s. we are offered fewer than $r/2$ such edges, meaning that we are offered at least $n \log n$ edges within $V \setminus W$. As long as each vertex in W remains isolated, every edge within $V \setminus W$ will be accepted.

Let G be the Erdős-Rényi random graph on the vertex set $V \setminus W$ with $n \log n$ random edges. By the above considerations, $G(\tau + r)\setminus W$ can be coupled with G in such a way that a.a.s. $G(\tau + r)\setminus W \supseteq G$. It is well known that G a.a.s. has a Hamilton cycle (see for example [24, Section 5.1]), which means that it has a perfect matching after deleting any vertex. The same is a.a.s. true for $G(\tau + r) \setminus W$, meaning that no edge involving an isolated vertex can ever be accepted after this point.

4.2 Cherries: the negative part

In this subsection we will prove that if $f = \omega(\sqrt{n}/\log n)$ and $f = o(n)$ then a.a.s. we accept some edge involving an isolated vertex, after time τ. This suffices, because the case where $f = \Omega(n)$ is handled by Proposition 4.

We say a path of length 2 in a graph is a *cherry* if its two endpoints have degree 1 in the graph. A matching can use at most two of the three vertices of a cherry, so if $G(\tau)$ has a cherry then there is some freedom to add an edge involving an isolated vertex, without creating a k-matching.

Lemma 20. *Suppose $f = \omega(\sqrt{n}/\log n)$. Let $g = f \log n/\sqrt{n} \to \infty$, and choose $h \to \infty$ to satisfy $h = o(\log g)$. Then a.a.s. $G\left(t^+_h\right)$ has a cherry.*

Proof. We use a two-phase argument similar to the proof of Lemma 15. Consider $G_1 \in \mathbb{G}(n, p_1)$ and $G_2 \in \mathbb{G}(n, p_2)$, with

$$p_1 = \frac{\log n - \log(f+1) - h}{n} = \frac{\log n/2 + \log\log n - (1+o(1))\log g}{n},$$

and $p_2 \le 5h/n$ chosen such that $p_2(1 - p_1) = 4h/n$. Then by the Chernoff bound, a.a.s.

$$\left|e(G_1) - (\log n - \log(f+1) - h)\frac{n}{2}\right| = o(n),$$
$$|e(G_2 \setminus G_1) - 2hn| = o(n),$$

implying that $e(G_1) < t_h^-$ and $e(G_1 \cup G_2) > t_h^+$. Conditioned on $e(G_1)$ (respectively $e(G_1 \cup G_2)$), note that G_1 (respectively $G_1 \cup G_2$) has an Erdős-Rényi random graph distribution, so we can couple G_1, G_2 and $(G(t))_t$ such that a.a.s. $G_1 \subseteq G^{\mathrm{all}}(t_h^-) = G(t_h^-)$ and $G_1 \cup G_2 \supseteq G^{\mathrm{all}}(t_h^+) \supseteq G(t_h^+)$. We will prove that a.a.s. G_1 has many cherries, and a.a.s. at least one of these cherries remains in $G_2 \cup G_1$.

Let X be the number of cherries in G_1. Then

$$\mathbb{E}X = n\binom{n-1}{2}p_1^2(1-p_1)^{2n-5} = \Omega\big(n^2p_1^2g^2/\log^2 n\big).$$

Considering separately the cases $g \geq \log^2 n$ and $g < \log^2 n$, noting that $np_1 \to \infty$, it follows that $\mathbb{E}X = \omega(g)$. Considering all the possible ways a pair of distinct cherries can intersect, we can compute

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \left(n\binom{n-1}{2}(n-3)\binom{n-4}{2}\right)p_1^4(1-p_1)^{4n-14} + O(n^5p_1^4(1-p_1)^{4n} + n^4p_1^3(1-p_1)^{3n}) \\ &= (1-o(1))(\mathbb{E}X)^2 + O(np_1e^{-np_1}\mathbb{E}X) = (1-o(1))(\mathbb{E}X)^2, \\ \operatorname{Var} X &= o\Big((\mathbb{E}X)^2\Big).\end{aligned}$$

Therefore a.a.s. G_1 has at least $\mathbb{E}X/2 > 2g$ cherries. We say a pair of cherries is "externally intersecting" if they intersect in their degree-1 vertices (the only way this can occur is if the union of the two is a 3-edge star). Let Z be the number of pairs of externally intersecting cherries; then we can compute

$$\mathbb{E}Z = O\Big(n^4p_1^3(1-p_1)^{3n}\Big) = O(np_1e^{-np_1}\mathbb{E}X) = o(\mathbb{E}X).$$

So, a.a.s. there is a collection of g cherries in G_1 which are pairwise externally disjoint. Condition on such an outcome of G_1. Let Y be the number of these cherries in this collection that remain in $G_1 \cup G_2$. Then, we can compute

$$\begin{aligned}\mathbb{E}Y &= g(1-p_2)^{2n-5} = \Omega\Big(ge^{-5h}\Big) = g^{1-o(1)} = \omega(1), \\ \mathbb{E}[Y(Y-1)] &= g(g-1)(1-p_2)^{4n-18} = (1-o(1))(\mathbb{E}Y)^2, \\ \operatorname{Var}(Y) &= o\big((\mathbb{E}X)^2\big).\end{aligned}$$

So, a.a.s. $Y > 0$. $\square$

Now, let $P_{G(\tau)}$ be the set of pairs consisting of an isolated vertex of $G(\tau)$ and a vertex of degree at least 2 in $G(\tau)$. Let h be as in Lemma 20. We claim that in order to prove that a.a.s. some edge involving an isolated vertex of $G(\tau)$ is accepted, it suffices to show that a.a.s. there is a time $t < t_h^+$ for which an element of $P_{G(\tau)}$ is offered as $e(t)$. Indeed, let $e(t)$ be such an element, and suppose that $G(t_h^+)$ has $f+1$ isolated vertices (otherwise we are done). By Lemma 20, a.a.s. $G(t_h^+) + e(t)$ has a cherry, and at most two of the three vertices of this cherry can be used in a matching. Since $G(t_h^+) + e(t)$ has $f = n - 2k$ isolated vertices, it is k-matching-free, so $e(t)$ is accepted at time t.

In view of the above discussion, condition on $\tau \leq t_{h/2}^+$ such that no edge of $P_{G(\tau)}$ has been offered yet. The probability we are offered an edge of $P_{G(\tau)}$ before time t_h^+ is at least

$$1 - \left(1 - \frac{|P_{G(\tau)}|}{\binom{n}{2}}\right)^{hn/2} \geq 1 - e^{-\Omega(|P_{G(\tau)}|h/n)},$$

so it actually suffices to show that a.a.s. $\left|P_{G(\tau)}\right| = \Omega(n)$. But this is an immediate consequence of the fact that a.a.s. $G(\tau)$ has $\Omega(n)$ vertices of degree at least 2. This can be proved by applying the second moment method to the number of isolated vertices and to the number of degree-1 vertices in $G(t_h^-)$, for sufficiently slowly-growing h such that $t_h^- = \omega(n)$.

4.3 Proof of Lemma 18

Let $g = \sqrt{n}/(f \log n) \to \infty$ and choose $h \to \infty$ to satisfy $h = o(\log g)$. Let $W(t)$ be the set of isolated vertices in $G^{\mathrm{all}}(t)$. We will prove that a.a.s. for each $t_h^- \leq t \leq t_h^+$, if $n - |W(t)|$ is odd then $G^{\mathrm{all}}(t) \setminus W(t)$ is matching-resilient.

To accomplish this, we adapt the method of Łuczak and Ruciński [29] used to study tree-packings. For the special case of matchings, this method was outlined in [24, Section 4.1]. Where possible, we will re-use lemmas from [29] and [24].

First, the following lemma follows directly from parts (i) and (ii) of [29, Theorem 3].

Lemma 21. *For any $h \to \infty$, a.a.s. for each $t \geq (\log n/2 + \log\log n + h)n/2$, the largest component of $G^{\mathrm{all}}(t)$ has no cherries.*

The next lemma follows directly from [29, Lemma 1].

Lemma 22. *For any $h \to \infty$, a.a.s. for each $t \geq (\log n/2 + \log\log n/2 + h)n/2$, in $G^{\mathrm{all}}(t)$ there are only isolated vertices outside the largest component.*

The next lemma is a slight adaptation of [24, Lemma 4.7] (essentially the only difference is that we need a certain property to hold for a range of $G^{\mathrm{all}}(t)$ instead of a single random graph).

Lemma 23. *For any $c > 0$, a.a.s. for all $n \log n/4 \leq t \leq n \log n$, every bipartite subgraph with minimum degree at least $c \log n$, induced in $G^{\mathrm{all}}(t)$ by two sets of equal size, contains a perfect matching.*

Proof. From [29, Lemma 4], we know that a.a.s. for all such t,

(1) for every pair of disjoint subsets of size $n(\log\log n)^2/\log n$, there is an edge between them in $G^{\mathrm{all}}(t)$,

(2) every set S of at most $2n(\log\log n)^2/\log n$ vertices induces fewer than $(\log\log n)^3|S|$ edges in $G^{\mathrm{all}}(t)$.

We can then conclude the proof exactly as in [24, Lemma 4.7], using Hall's theorem. □

The next lemma follows from [29, Lemma 4]. Let $A = \{1, \ldots, \lfloor n/2 \rfloor\}$ and $B = \{\lfloor n/2 \rfloor + 1, \ldots, n\}$, so that A and B give a fixed balanced partition of $[n]$. As in [29, Section 3], say a vertex is *bad* if it has fewer than $\log n/200$ neighbours in A or in B. Say a vertex is *small* if $d(v) < 40$.

Lemma 24. *A.a.s. for all $n \log n/4 \leq t \leq n \log n$, the following properties hold.*

(1) $G^{\mathrm{all}}(t)$ has no more than $n/\log^{40} n$ bad vertices,

(2) $G^{\mathrm{all}}(t)$ has no 8 bad vertices within distance 20 from each other,

(3) $G^{\mathrm{all}}(t)$ has no 2 small and 1 bad vertices within distance 20 from each other,

(4) $G^{\mathrm{all}}(t)$ has maximum degree less than $6 \log n$.

Now we can prove Lemma 18. We closely follow the proof of [24, Theorem 4.4].

Proof of Lemma 18. A.a.s. For any $t_h^- \le t \le t_h^+$ the graph $G^{\text{all}}(t)$ satisfies the properties of Lemmas 21 to 24, so we assume these properties hold for the remainder of the proof. Consider any $t_h^- \le t \le t_h^+$, suppose $n - |W(t)|$ is odd, and let $G = G^{\text{all}}(t) \setminus W(t)$. Let v be any non-dangerous vertex. We will show that $G \setminus v$ has a perfect matching.

Order the bad vertices of G by degrees:

$$d(v_1) \le d(v_2) \le \cdots \le d(v_\ell).$$

We greedily match these vertices one-by-one with vertices $u_1, \dots, u_\ell$, as follows. Suppose $v_1, \dots, v_{i-1}$ are already matched with $u_1, \dots, u_{i-1}$ (some v_j may be matched with some v_q, which means $u_j = v_q$ and $u_q = v_j$). Let $V_{i-1} = \{v_1, \dots, v_{i-1}\}$ and $U_{i-1} = \{u_1, \dots, u_{i-1}\}$. Then, choose u_i to be an (arbitrary) vertex of smallest degree among $N_G(v_i) \setminus (V_{i-1} \cup U_{i-1} \cup \{v\})$. We need to show that this set is always nonempty, so that this choice is always possible.

- If $d(v_i) = 1$ then u_i is the unique neighbour of v_i. By definition u_i is dangerous, so $u_i \ne v$. Since G has no component of size 2, u_i does not have degree 1. Since the v_i are ordered by degrees, this means $u_i \notin V_{i-1}$. Since G has no cherry, $u_i \notin U_{i-1}$.
- Suppose $d(v_i) = 2$. Since there are no three small vertices within distance 4 of each other, there is at most one element of V_{i-1} within distance 2 of v_i. We consider three cases.
  - If v_i has no neighbour in $V_{i-1} \cup U_{i-1}$, then v_i has at least one neighbour other than v, which is a viable choice for u_i.
  - If v_i has exactly one neighbour in $V_{i-1} \cup U_{i-1}$, then some vertex in V_{i-1} is within distance 2 of v_i. By definition the other neighbour of v_i is dangerous, so it cannot have been chosen for v and is a viable choice for u_i.
  - The remaining case is that v_i has two neighbours in $V_{i-1} \cup U_{i-1}$. These neighbours cannot be of the form $\{v_j, v_q\}$ or $\{v_j, u_q\}$ or $\{u_j, u_q\}$ for $j \ne q$, as this would imply two different elements of V_{i-1} within distance 2 of v_i. So, it actually remains to consider the case $N_G(v_i) = \{v_j, u_j\}$ for some $j < i$. This would mean that v_i was a viable choice for u_j but was not chosen, which means $d(u_j) \le 2$. But this would give three small vertices at distance 1 of each other, which is impossible.
- If $3 \le d(v_i) \le 40$ then by the same reasoning as above, v_i has at most one neighbour in $V_{i-1} \cup U_{i-1}$. It's possible that v is also a neighbour of v_i, but there is still at least one neighbour left for u_i.
- If $d(v_i) \ge 41$, then v_i has at most 13 neighbours in $V_{i-1} \cup U_{i-1}$, since otherwise there would be 7 bad vertices within distance 2 of v_i (and therefore 8 bad vertices within distance 4 of each other). So there are plenty of neighbours left for u_i.

We have proved that all bad vertices can be matched. After removing all vertices matched so far (and v), every vertex has at least $\log n/200 - 17$ neighbours in A and in B (no neighbourhood loses more than 17 vertices, or else there would be more than 8 bad vertices within distance 4 of each other). The remaining vertices in A and B may no longer form a balanced bipartition. In order to apply the property in Lemma 23, we move some carefully chosen vertices across the partition to balance it. A *2-independent set* of vertices is a set of vertices such that no two share a common neighbour. Recalling that G has maximum degree at most $6 \log n$, it has a 2-independent set of size

$\Omega(n/\log^2 n)$, which is more than the $O(n/\log^{40} n)$ vertices we must move to balance the bipartition. So, move a 2-independent set of vertices to balance the partition. At most one neighbour of each vertex is moved, so the edges between the parts form a balanced bipartite graph with minimum degree at least $\log n/200 - 18$. We can then apply the property in Lemma 23 to see that this graph has a perfect matching, finishing the proof. $\square$

5 Proof of Proposition 3

In this section we prove Proposition 3. Note that we may assume $k = \omega(\sqrt{n})$ (otherwise we can defer to Theorem 1), and we may assume that $k/n \leq 1/100$ (because if $k/n = \Omega(1)$ then Proposition 3 is trivial). Also, note that it actually suffices to prove that a.a.s. $G(N)$ has $k - f$ vertices of degree $n-1$, for some $f = O(k^2/n)$. Indeed, since $G(N)$ is k-matching-free, within the remaining $n-k+f$ vertices there is no matching of size f, meaning that there must be an independent set of size $n-k+f-2(f-1) > n-k-f$.

Define τ as in Section 2.1. We first need a stronger version of Lemma 5.

Lemma 25. *A.a.s.* $\tau \leq (1 + O(k/n))k$.

Proof. Let $q = (1 + 11k/n)k$. We proceed in basically the same way as Lemma 5. For each $t \leq q$, $G^{\mathrm{all}}(t-1)$ has at most q edges, comprising at most $2k$ vertices. The probability that $e(t)$ does not increase the matching number is at most the probability that it intersects the edges of $G^{\mathrm{all}}(t-1)$, which is at most $2qn/\left(\binom{n}{2} - q\right) \leq 9k/n$. The number of such steps is stochastically dominated by the binomial distribution $\mathrm{Bin}(q, 9k/n)$, which has expectation $9qk/n \leq 10k^2/n$. So, recalling the assumption that $k^2/n = \omega(1)$, by the Chernoff bound a.a.s. there are at most $11k^2/n$ steps among the first q that do not increase the matching number. It follows that $\tau \leq q$. $\square$

Now, condition on an outcome of $\tau, e(1), \ldots, e(\tau)$ such that $\tau \leq (1 + O(k/n))k$, and recall the definitions of M, A, B and augmenting paths from Section 2.1. In $G^{\mathrm{all}}(\tau)$, note that there are only $O(k^2/n)$ edges not in M, and each such edge can be incident to at most two edges of M. Therefore, $(1 - O(k/n))k$ of the $k-1$ edges of M are isolated in $G^{\mathrm{all}}(\tau)$. Let M' be the sub-matching of such edges. For each $e \in M'$, let E_e be the event that we are offered an edge between some $b \in B$ and some endpoint a_e^+ of e (let a_e^- be the other endpoint of e) before we are ever offered any edges between e and A, and then we are offered another edge between a_e^+ and B before we are offered the edge $\{a_e^-, b\}$ or any further edges between e and A. There are $2|B|$ possible edges between e and B, out of $2(n-2)$ possible edges involving e other than e itself. After $\{a_e^+, b\}$ has been revealed, there are $|B| - 1$ possible further edges between a_e^+ and B, out of a total of $n-3$ possible further edges involving a_e^+. There are also $|A| - 2$ edges between a_e^- and A. So,

$$\Pr(E_e) = \frac{2|B|}{2(n-2)} \cdot \frac{|B| - 1}{(n-3) + 1 + (|A| - 2)} = 1 - O\left(\frac{k}{n}\right). \tag{2}$$

Now, consider distinct $e, e' \in M'$. We will show that E_e and $E_{e'}$ are essentially independent. For a possible edge $f \subseteq A$, let Q_f^e be the event that f is not offered until three edges have already been offered between e and B (this means that f is not offered until E_e has already been determined). Recalling $|B| = \Theta(n)$, we have

$$\Pr(Q_f^e) = \frac{2|B|}{2|B|+1} \cdot \frac{2|B|-1}{2|B|} \cdot \frac{2|B|-2}{2|B|-1} = 1 - O\left(\frac{1}{n}\right).$$

Now, let Q be the intersection of the events $Q_f^e \cap Q_f^{e'}$ for each of the four possible edges f between e and e'. If Q holds, then none of these four edges are offered until E_e and $E_{e'}$ have already been determined. By the union bound, $\Pr(Q) = 1 - O(1/n)$. Let E'_e and $E'_{e'}$ have the same definitions as the events E_e and $E_{e'}$, but ignoring all edges between e and e'. This means that E'_e and $E'_{e'}$ are independent. Now, note that

$$\Pr(E_e \cap E_{e'}) = O(\Pr(\overline{Q})) + \Pr(E_e \cap E_{e'} \cap Q) = O\left(\frac{1}{n}\right) + \Pr(E_e \cap E_{e'} \cap Q),$$

and similarly $\Pr(E'_e \cap E'_{e'}) = O(1/n) + \Pr(E'_e \cap E'_{e'} \cap Q)$. Observe that $E_e \cap E_{e'} \cap Q$ is actually the same event as $E'_e \cap E'_{e'} \cap Q$, so

$$\Pr(E_e \cap E_{e'}) = \Pr(E'_e \cap E'_{e'}) + O\left(\frac{1}{n}\right) = \Pr(E'_e)\Pr(E'_{e'}) + O\left(\frac{1}{n}\right) = \Pr(E_e)^2 + O\left(\frac{1}{n}\right).$$

Let X be the number of edges $e \in M'$ such that E_e holds. By the above calculations, $\mathbb{E}X = k - O(k^2/n)$ and

$$\operatorname{Var} X = \sum_{(e,e') \in E(M')^2} (\Pr(E_e \cap E_{e'}) - \Pr(E_e)\Pr(E_{e'})) = O(k^2/n).$$

Recalling the assumption that $k^2/n = \omega(1)$, a.a.s. $X \geq \mathbb{E}X - k^2/n$ by Chebyshev's inequality, implying that E_e holds for $(1 - O(k/n))k$ edges e of M'. Now, the following lemma completes the proof.

Lemma 26. *If E_e holds, then a_e^+ has degree $n-1$ in $G(N)$.*

Proof. If E_e holds, the first edge between a_e^+ and B will be accepted, all subsequent edges between the other endpoint a_e^- of e and B will be rejected (because they would create a length-3 augmenting path), and the next edge between a_e^+ and B will be accepted (at this point the connected component of e will then be a 3-edge star, involving two vertices $b_1, b_2 \in B$). Now, suppose for the purpose of contradiction that some further edge $e(t)$ involving a_e^+ is rejected. This means that $e(t)$ would introduce an augmenting path P starting at some vertex in B, passing through e and $e(t)$ consecutively, then ending at some vertex b^* in B. Without loss of generality suppose $b_1 \neq b^*$; but then $G(t-1)$ already had an augmenting path, obtained by replacing the portion of P after e by the edge $\{a_e^+, b_1\}$. This is a contradiction. ◻

6 Proof of Proposition 4

First, claim (1) will be an immediate consequence of two simple lemmas.

Lemma 27. *A.a.s. $G \in \mathbb{G}(n, 6n)$ has no independent set of size $n/3$.*

Proof. Let $p = 6n/\binom{n}{2}$; using say [24, Corollary 1.16] it suffices to show that $G' \in \mathbb{G}(n,p)$ has the required property. Let X be the number of independent sets of size $n/3$ in G'; then using Stirling's approximation we have

$$\mathbb{E}X = \binom{n}{n/3}(1-p)^{\binom{n/3}{2}} \leq \exp\left(-\left(\frac{1}{3}\log\frac{1}{3} + \frac{2}{3}\log\frac{2}{3} + \frac{2}{3} + o(1)\right)n\right) = o(1),$$

so the desired result follows from Markov's inequality. ◻

Lemma 28. *A.a.s. the largest matching in $G \in \mathbb{G}(n, 6n)$ has size at most $n/2 - e^{-13}n/2$.*

Proof. Let $p = 6n/\binom{n}{2}$; it suffices to show that $G' \in \mathbb{G}(n, p)$ has the required property. Let X be the number of isolated vertices in G'; then we compute $\mathbb{E}X = n(1-p)^{n-1} = (1 - o(1))e^{-12}n = \Omega(n)$, whereas one can check that $\operatorname{Var} X = o(n^2) = o\Big((\mathbb{E}X)^2\Big)$. So, a.a.s. G' has $e^{-13}n$ isolated vertices, which cannot contribute to a matching. $\square$

Recall the definition of τ from Sections 2 and 5; we have proved that for $k \geq n/2 - e^{-13}n/2$, a.a.s. $G^{\text{all}}(\tau) = G(\tau) \subseteq G(N)$ has no independent set of size $n/3 = n - k - \Omega(n)$, proving (1).

Now we consider claim (2). Say a 2-path in a graph is *isolated* if there are no edges between the 2-path and the rest of the graph.

Lemma 29. *Consider any constants $0 < R_1 < R_2$. Then a.a.s. each $G^{\text{all}}(t)$, for $R_1 n \leq t \leq R_2 n$, has $\Omega(n)$ isolated 2-paths.*

Proof. Consider $R_1 n \leq t \leq R_2 n$. To use the union bound, it suffices to show that $G^{\text{all}}(t)$ has the required property with probability $1 - o(1/n)$. Let $p = t/\binom{n}{2}$; by Pittel's inequality (see [24, Section 1.4]), it actually suffices to show this for $G' \in \mathbb{G}(n, p)$, instead. Let X be the number of isolated 2-paths in G', so that

$$\mathbb{E}X = 3\binom{n}{3}p^2(1-p)^{3(n-3)} = \Omega(n).$$

Observe that changing the status of any edge changes the value of X by at most 2, so by a Bernstein-type concentration inequality (see for example [28, Theorem 2.11]),

$$\Pr(X \leq \mathbb{E}X/2) \leq \exp\left(-\frac{(\mathbb{E}X/2)^2}{16t + 2\mathbb{E}X}\right) = e^{-\Omega(n)},$$

as desired. $\square$

Let $R_1 = (k-1)/n = \Theta(1)$, and again recall the definition of τ from Sections 2 and 5. Using Lemma 14, we know that there is R_2 such that a.a.s. $\tau \leq R_2 n$, and trivially (as remarked in Section 2) we have $\tau \geq k - 1 = R_1 n$. It follows that that a.a.s. $G^{\text{all}}(\tau) = G(\tau)$ has $\Omega(n)$ isolated 2-paths. For any maximum $(k-1)$-matching in $G(\tau)$, there is at least one vertex in every isolated 2-path which does not contribute to that matching, so there are a.a.s. $2(k-1) + \Omega(n)$ non-isolated vertices. This proves claim (2).

7 Concluding remarks

In this paper we studied the random greedy k-matching-free process, in which edges are iteratively added to an empty graph, each chosen uniformly at random subject to the restriction that no k-matching is formed. We discovered that if $k = o(n)$ or $n - 2k = o(\sqrt{n}/\log n)$ then this process is likely to produce an extremal k-matching-free graph. We also made a first step exploring the intermediate regime, but here there is much more work to be done. In particular, Proposition 4 says that there is a range of values of k for which the outcome of the k-matching-free process is likely to be far from an extremal graph; we wonder whether these random graphs have interesting properties that may be useful for other problems.

We also hope that the ideas in this paper may be useful for studying other related kinds of random processes. For example, it is natural to ask about the k-path-free process, or the restricted-girth process where we greedily add edges keeping the girth above some value k. (The restricted-girth process has already been studied for fixed k by Osthus and Taraz [30]; see also the work of Bayati, Montanari and Saberi [1] on a slightly different process.) There are also natural generalisations of these processes to hypergraphs. In particular, define the (-2)-*girth* of a 3-uniform hypergraph to be the smallest integer $g \geq 4$ such that there is a set of g vertices spanning at least $g-2$ edges. Erdős [13] asked in 1973 whether there are hypergraphs with large girth and quadratically many edges; in an earlier version of this paper we suggested that analysis of a hypergraph generalisation of the restricted-girth process might lead to progress on this question. Since that time, Glock, Kühn, Lo and Osthus [21] managed to prove Erdős' conjecture in precisely this way. (The authors also mention that the same result was independently proved by Bohman and Warnke).

Finally, we remarked in the introduction that despite the matching number and vertex cover number of a graph being very closely related to each other, the restricted covering process exhibits quite trivial behaviour compared to the matching-free process. Perhaps it would be interesting to explore more closely the relationship between these two parameters by considering random graph models where the vertex cover number and matching number are constrained to be equal (for example, one could consider a random process where edges are added as long as they do not separate the vertex cover number and matching number).

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